Assignment 6

Due Tuesday, Dec 6, 05, at 10:00 AM in Class

1. A state wants to produce license plates consisting of 4 uppercase letters, a space, and three digits (zero is excluded). If repetition is allowed (i.e. you can repeat digits/letters), how many different license plates are possible?

   Solution: There are $(26)^4$ ways to choose 4 uppercase letters and 9^3 ways to choose 3 nonzero digits. Thus, by the multiplication principle, there are $(26)^4 \cdot 9^3$ possible different license plates.

2. An instructor has divided his class into 8 groups. Each group has to give a presentation. The instructor wants the presentations to be given in the last 3 classes of the semester. He wants 3 presentations each day except the last day in which he wants only two presentations. In how many ways can this be done?

   Solution: This is exactly as the question: In how many ways can 8 people wait in a line? Or in how many ways can we put 8 distinct balls in 8 distinct boxes at most one ball a box? So, the answer is 8! (or $P(8,8)$).

3. A systems administrator decided to make a password consist of 7 characters, the first has to be from the set $\{A, B, C, D\}$ and the remaining 6 characters can be either lowercase English alphabets or digits. How many different passwords are possible?

   Solution: We can first choose the first character. There are 4 possible ways to do that. Then choose the remaining 6 characters. There are $(26+10)^6$ possible ways to do that. Thus, by the multiplication principle, the answer is $4 \cdot (36)^6$.

4. In how many ways can you arrange the letters of the word CONNECTICUT (i.e. how many distinct permutations of the letters of CONNECTICUT are there)?

   Answer: $\frac{11!}{3! \cdot 2! \cdot 2!}$.

5. (a) (*Original*) In how many ways can you arrange the letters of the word CONNECTICUT if now two C's are consecutive?

   Answer: The asnwer to (6) + the answer to (7).

   (b) (*What I wanted*) In how many ways can you arrange the letters of the word CONNECTICUT if no two C's are consecutive? Answer: $\frac{8!}{2! \cdot 2!} C(9,3)$.

6. In how many ways can you arrange the letters of the word CONNECTICUT if the three C's must be consecutive?

   Answer: $\frac{9!}{2! \cdot 2!}$.

7. In how many ways can you arrange the letters of the word CONNECTICUT if two of the C's must be together and the third is separate?

   Answer: $8 \cdot \frac{9!}{2! \cdot 2!}$.

8. A committee consists of 4 women and 6 men has to be chosen from a group of 20 women and 30 men. How many different committees can be chosen?

   Solution: First choose the women. There are $C(20, 4)$ ways to do that. Then choose the men. There are $C(30, 6)$ ways to do that. Thus, by the multiplication principle, the answer is $C(20, 4) \cdot C(30, 6)$.

9. In how many ways can a person choose 4 CDs from the top ten list if repetition is allowed and if there are at least 4 CDs from each one of the top ten?

   Solution: This is the same as asking: In how many ways can you put 4 identical balls into 10 distinct boxes any number in a box? The answer is: $C(10+4-1, 4)$.

10. In how many ways can the letters of the English alphabet be arranged so that there are exactly 10 letters between a and z?

    Solution: There $P(24, 10)$ ways of choosing the 10 letters between a and z. Now look at those 10 letters and at a and z as one object. Then, you have now 14+1 abjects to arrange. The number of different arrangements of them is 15!. But, also either a will be first or z will be first, so you have to multiply by 2. Thus, by the multiplication principle, the answer is: $2 \cdot P(24, 10) \cdot 15!$.

11. A man, a woman, a boy, a girl, a dog, and a cat, are walking down a road one after another. In how many ways can this happen if the dog has to be between the man and the boy?

    Solution: Think of the man, the boy, and the dog, as one object. Thus, you have 4 objects to order. You can do that in 4! different ways. Now since either the boy will be first or the man will be first, then you'll have to multiply by 2. Thus, the answer is: $2 \cdot 4!$.

12. In how many ways can 8 books be split among Jay, Mary, and Chris, if Jay has to get 4 books, and Mary and Chris each gets two books?

    Solution: The question is the same as: In how many ways can you arrange the letters of the word JJJJMMCC? So, the answer is $\frac{8!}{4! \cdot 2! \cdot 2!}$.

13. In how many ways can you distribute 6 distinct red balls and 8 identical blue balls on 30 distinct boxes at most one ball to a box?

    Answer: $P(30, 6) \cdot C(24, 8)$ or $C(30, 8) \cdot P(22, 6)$ or $\frac{30!}{8!}$.

14. In a small college, there are 250 students taking math, 310 taking CS, 130 taking both math and CS. How many are taking math or CS but not both?

    Answer: $250 + 310 - 2(130) = 300$.

15. In how many ways can you arrange 18 distinct CS books and 15 distinct math books on a shelf if the math books are to be together?

    Answer: $19! \cdot 15!$.

16. In how many ways can you put 7 identical red balls into 15 distinct boxes (here there is no limit on the number of balls in a box).

    Answer: $C(15 + 7 - 1, 7)$.

17. 90 people are standing in a line. In how many ways, can they change their positions so that exactly 25 of them keep their original positions?

    Answer: $C(90, 25)D_{90-25}$.

18. 90 people are standing in a line. In how many ways, can they change their positions so that at most 25 of them keep their original positions?

    Answer: $\sum_{i=0}^{25} C(90, i)D_{90-i}$.

19. How large a group should be to ensure at least 7 people in the group were born in the same month?

    Answer: 73.

20. A seven-person committee that includes Mary and Chris is to select a chairperson, a secretary, and a treasurer. In how many ways can this be done if Mary is to hold one and only one of the offices?

    Answer: $3 \cdot P(6, 2)$.

21. (a) (*Original*) How many permutations of the letters ABCDEFG contain the substring CDE?

       Answer: $5!$.

    (b) (*What I wanted*) How many permutations of the letters ABCDEFG contain a permutation of the substring CDE?

       Answer: $5! \cdot 3!$.

22. How many 7-digit decimal numbers are there if repetition is not allowed?

    Answer: $9 \cdot 9 \cdot 8 \cdot 7 \cdot 6 \cdot 5 \cdot 4$.

23. How many 7-digit decimal numbers are there if the degit has to end (from the right) with a 6 and if repetition is not allowed?

    Answer: $8 \cdot 8 \cdot 7 \cdot 6 \cdot 5 \cdot 4 \cdot 1$.

24. In how many ways can you put 3 distinct red balls and 4 identical blue balls in 20 distinct boxes if the box can contain at most one ball?

Answer: $P(20,3) \cdot C(17,4)$ or $C(20,4) \cdot P(16,3)$ or $\frac{20!}{4!}$.

25. In how many ways can you give 30 hats to 30 people if exactly 11 of them are to receive their original hats?

Answer: $C(30,11)D_{30-11}$.

26. In how many ways can you give 30 hats to 30 people if at most 11 of them are to receive their original hats?

Answer: $\sum_{i=0}^{11} C(30,i)D_{30-i}$.

27. In how many ways can you give 30 hats to 30 people if at least 11 of them are to receive their original hats?

Answer: $\sum_{i=11}^{30} C(30,i)D_{30-i}$.

28. In how many ways can 40 indistinguishable fish be put in 8 ponds if each pond must contain at least 3 fish?

Answer: $C(16+8-1,16)$.

29. Find the coefficient of $x^{50}y^{75}$ in the binomial expansion of $(3x^2-5y)^{100}$.

Answer: $C(100,25) \cdot 3^{25} \cdot (-5)^{75}$.

30. Find the coefficient of $x^{50}y^{76}$ in the binomial expansion of $(3x^2-5y)^{100}$.

Answer: 0.

31. Find the coefficient of x^{50} in the binomial expansion of $(3x^4+\frac{5}{x})^{100}$.

Answer: $C(100,30) \cdot 3^{30} \cdot (5)^{70}$.

32. Find the coefficient of x^3 in the binomial expansion of $(3x^4+\frac{5}{x})^{100}$.

Answer: 0.

33. Find the middle term of the binomial expansion of $(3x^7+5)^{30}$.

Answer: $C(30,15) \cdot 3^{15} \cdot (5)^{15}$.