CSIT 241 Binary Relations

Definition: Let A and B be nonempty sets. A *binary relation* from A to B is a subset of $A \times B$.

Definition: Let A be a nonempty set. A *binary relation* on A is a subset of $A \times A$.

Remarks:

(1) If (x, y) belongs to a relation R, then we write xRy. Note that if xRy, then it is not necessarily that yRx.

(2) Every binary relation on a set A can be represented by a matrix.

(3) Every binary relation on a set A can be represented by a directed graph.

(4) Every function is a binary relation, but not every binary relation is a function.

Note that a binary relation on A simply defines a relation between the elements of A.

Example: Let A be the set of all residents of San Fransisco. Define the following relation M on A:

$$M = \{(x, y) \mid \text{x is the wife of y}\}.$$

Example: Let A be the set of all residents of San Fransisco. Define the following relation *Couples* on A:

$$Couples = \{(x, y) \mid \text{x is married to y}\}.$$

Example: Let A be the set of all humans. Define the following relation $EyeColor$ on A:

$$EyeColor = \{(x, y) \mid \text{x has the same eye color as y}\}.$$

Example: Let A be the set of SUNY Fredonia's students and let B be the set of courses offered by SUNY Fredonia. Define the following relation $SameCourse$ from A to B:

$$SameCourse = \{(s, c) \mid \text{s is taking c}\}.$$

Example: Let $A = \mathbb{R}$. Define the following relation S on A:

$$S = \{(x, y) \in \mathbb{R} \times \mathbb{R} \mid x \leq y\}.$$

Example: Let $A = \mathbb{Z}$. Define the following relation T on A:

$$T = \{(x, y) \in \mathbb{Z} \times \mathbb{Z} \mid x^2 = y^2\}.$$

Reamrks: Binary relations can be represented as digraphs and as matrices.

Definitions: Let R be a relation on a set A. R is said to be

1. *Reflexive* if $(a, a) \in R$, for all $a \in A$.
2. *Symmetric* if $(a, b) \in R$ implies $(b, a) \in R$, for all a and b in A.
3. *Transitive* if $(a, b) \in R$ and $(b, c) \in R$, implies $(a, c) \in R$, for all a, b, and c in A.
4. *Antisymmetric* if $(a, b) \in R$ and $(b, a) \in R$ implies $a = b$, for all a and b in A. I.e. if $a \neq b$ and $(a, b) \in R$, then (b, a) cannot be in R. If $b = a$, then there is no problem.

5. *An equivelence relation* if R is reflexive, symmetric, and transitive.
6. *A partial order* if R is reflexive, antisymmetric, and transitive.
7. Asymmetric if whenever $(a, b) \in R$, then $(b, a) \notin R$, for all a and b in A.
8. *Circular* if $(a, b) \in R$ and $(b, c) \in R$, implies $(c, a) \in R$, for all a, b, and c in A.

Notation: If R is a partial order on A, we denote R by $\leq$ and we write $(A, \leq)$ is a *poset*.

Definition: Let R be a partial order on set A and let x and y be two elements in A. x and y are said to be *comparable* if either $(x, y) \in R$ or $(y, x) \in R$. If x and y are not comparable, they are called *incomparable*.

Definition: Let R be a partial order on set A. R is called a *total order* if every two elements of A are comparable.

Example: The binary relation S defined above is a total order, while the binary relation R defined on $\mathbb{Z}^+$ by xRy if and only if $x|y$ is a partial order but not a total order.

Example: The relation S defined above is reflexive, because every real number is less than or equal to itself. Mathematically, $x \leq x$, for all $x \in \mathbb{R}$. Thus, $(x, x) \in S$, for all $x \in \mathbb{R}$.

S is not symmetric. For example, $(2, 3) \in S$, because $2 \leq 3$. But, $(3, 2)$ is not in S, because 3 is not less than or equal to 2.

S is transitive because if $(x, y) \in S$ and $(y, z) \in S$, then $(x, z) \in S$. Because $(x, y) \in S$ and $(y, z) \in S$ mean $x \leq y$ and $y \leq z$. Thus, $x \leq y \leq z$. Therefore, $x \leq z$, which implies $(x, z) \in S$.

S is antisymmetric because if $(x, y) \in S$ and $(y, x) \in S$, then $x \leq y$ and $y \leq x$. Thus, $x \leq y \leq x$. This implies $y = x$.

Thus, S is a partial order but not an equivalence relation.

Example: The relation T defined above is reflexive because every real number is equal to the square of itself. Mathematically, $x^2 = x^2$, for all $x \in \mathbb{Z}$. Thus, $(x, x) \in T$, for all $x \in \mathbb{Z}$.

T is symmetric, because if $(x, y) \in T$, then $x^2 = y^2$. This implies $y^2 = x^2$, which implies $(y, x) \in T$.

T is transitive because if $(x, y) \in T$ and $(y, z) \in T$, then $(x, z) \in T$. Because $(x, y) \in T$ and $(y, z) \in T$ mean $x^2 = y^2$ and $y^2 = z^2$. Thus, $x^2 = y^2 = z^2$. Therefore, $x^2 = z^2$, which implies $(x, z) \in T$.

T is not antisymmetric. For example, $(-2, 2) \in T$ and $(2, -2) \in T$, but $2 \neq -2$.

Thus, T is an equivalence relation but not a partial order.

Let R be the relation on $\mathbb{Z}$ defined by:

$$aRb \text{ iff } 3a + b \text{ is a multiple of} 4.$$

Is R an equivalence relation on $\mathbb{Z}$?

Solution: R is an equivalence relation, because

(1) R is reflexive, because $\forall a \in Z$, $3a+a = 4a$ which is divisible by 4. Thus, $(a, a) \in \mathbb{R}$.

(2) R is symmetric, because if $(a, b) \in R$, then $3a + b = 4m$, for some $m \in \mathbb{Z}$. Thus, $b = 4m - 3a$. But now $3b + a = 3(4m-3a)+a = 4(3m-2a)$. Notice that $3m-2a$ is an integer. Thus, $3b + a$ is divisible by 4, which means that $(b, a) \in R$.

(3) R is transitive, because if (a, b) and (b, c) are in R, then $3a + b = 4m$ for some $m \in \mathbb{Z}$ and $3b + c = 4k$ for some $k \in \mathbb{Z}$. Therefore, $3a + c = 4m - b + 4k - 3b = 4(m + k - b)$. Notice that $m + k - b$ is an integer. Thus, $3a + c$ is divisible by 4, which means $(a, c) \in R$.

Definition: Let R be an equivalence relation on a set A and let $a \in A$. The *equivalence class* of a is defined to be all elements of A that are related to a. The equivalence class of a is denoted $[a]$ or $\overline{a}$. Mathematically,

$$[a] = \{b \in A \mid (a, b) \in R\}.$$

Note: To find $[a]$ in R, look for all ordered pairs in R that start with a.

Definition: Let A and B be sets. A and B are said to be disjoint if $A \cap B = \phi$.

Definition: Let A_1, A_2, $\cdots$, A_n be nonempty sets. A_1, A_2, $\cdots$, A_n are said to be **pairwise disjoint** if the intersection of every two distinct ones of them is empty.

Definition: Let A_1, A_2, $\cdots$, A_n be nonempty subsets of A. A_1, A_2, $\cdots$, A_n are said to **partition** A if they are pairwise disjoint and if $A_1 \cup A_2 \cup \cdots \cup A_n = A$.

Facts:

(1) Let $\mathcal{R}$ be an equivalence relation on a set A. Any two equivalence classes of $\mathcal{R}$ are either equal or disjoint. The union of all equivalence classes is equal to A and the distinct equivalence classes partition A.

(2) If $(a, b) \in R$, then $[a] = [b]$.

Remarks:

(1) Equivalence classes are defined only for equivalence relations.

(2) Let $\mathcal{R}$ be a partial order on A, then $\mathcal{R}$ establishes an order among the elements of A. One way to represent partial orders is Hasse diagrams.

Remark: Symmetric and antisymmetric are not opposite of each other, which means not symmetric does not mean antisymmetric and not antisymmetric does not mean symmetric. This means a relation can be both symmetric and antisymmetric at the same time. For example, if $A = \{1, 2, 3\}$ and $R_1 = \{(2, 2)\}$. Then R_1 is both symmetric and antisymmetric. R_1 is also transitive, but it is not reflexive because (for example) $(3, 3)$ is not in R_1. So, R_1 is neither an equivalence relation nor a partial order. Now let $R_2 = \{(1, 1), (2, 2), (3, 3)\}$ be a relation on the set A defined above. Then R_2 is an equivalnce relation and a partial order. Actually, R_2 is the only relation on A which is both an equivalence relation and a partial order.

Example 0.1 *Let* $A = \{1, 2, 3\}$ *and* $\mathcal{R} = \{(1,1),(2,2),(3,3),(1,2),(2,1)\}$. *Then* $\mathcal{R}$ *is an equivalnce relation on* A. *The equivalence classes are:*

[1]={ *all elements related to 1* }=$\{1, 2\}$.

[2]={ *all elements related to 2* }=$\{2, 1\}$.

So, $[2] = [1]$. *Remember the order within the set is not important.*

[3]={ *all elements related to 3* }=$\{3\}$.

Notice that the equivalence class of 3 is disjoint from the equivalence class of 1 and from the equivalence class of 2. Notice also that the union of the equivalence classes is equal to A.

Question: **Is it possible to have a binary relation which is both equivalence and a partial order? If yes, then give an example and explain the nature of such a binary relation.**

Answer: Let A be any nonempty set and let R be a binary relation on A. In order for R to be both equivalence and a partial order, we need R to be both reflexive, transitive, symmetric, and antisymmetric. But, in order for R to be antisymmetric, if it has an element of the form (a, b), and if $a \neq b$, then it cannot have (b, a). But, if (a, b) is in R and (b, a) is not, then R becomes not symmetric. Thus, R cannot have elements of the form (a, b) if $a \neq b$. Thus, the only elements which R can have are those of the form (a, a). But, in order for R to be reflexive, we need (a, a) to be in R for every element $a \in A$. (Notice that we have not used transitivity.) For example, if $A = \{1, 2, 3\}$, then the only equivalence relation and partial order on A is $\{(1, 1), (2, 2), (3, 3)\}$.

Composition of Binary Relations

Definition 0.2 *Let R_1 be a binary relation from X to Y and R_2 a binary relation from Y to Z; then*

$$R_2 o R_1 = \{(x, z) \mid (x, y) \in R_1 \text{ and } (y, z) \in R_2 \text{ for some } y \in Y\}.$$

Definition 0.3 *Let R be a relation on X; then*

$$R^{-1} = \{(x, y) \mid (y, x) \in R\}.$$

Question: Let R_1 be a binary relation from X to Y and R_2 a binary relation from Y to Z. Prove or disprove: $R_1 o R_2 = R_2 o R_1$.

Solution: The satement is false. You can come up with a counterexample easily. Do not forget that

$R_1 o R_2$ may be even undefined.

Remark 0.1 *Notice that every function is a relation. In other words, functions are special cases of relations. For example, if f is the function from $X = \{-1, 0, 2\}$ onto $Y = \{0, 1, 4\}$ given by $f(x) = x^2$. Then f can be represented as $f = \{(-1, 1), (0, 0), (4, 4)\}$.*

Notice also that if f is a function from X into X and if f^{-1} exists, then $fof^{-1}(x) = f^{-1}of(x) = x$, $\forall x \in X$. As a relation, we have $fof^{-1} = f^{-1}of = \{(x, x) \mid x \in X\}$. The question now is: does the same thing hold for any relation R on a set X? If no, then what are the properties of RoR^{-1}? We'll address some of those properties shortly.

Question: Let R be a relation on X. Prove or disprove:

(1) $RoR^{-1} = R^{-1}oR$.

(2) $|RoR^{-1}| = |R^{-1}oR|$.

(3) RoR^{-1} and $R^{-1}oR$ are both symmetric.

(4) If R is symmetric, then $R^{-1} = R$.

(5) $RoR = R$.

Solution:

(1) False. Counterexample: take $X = \{1, 2, 3\}$ and $R = \{(1, 3)\}$. Then $R^{-1} = \{(3, 1)\}$. Hence, $RoR^{-1} = \{(3, 3)\}$ and $R^{-1}oR = \{(1, 1)\}$.

(2) False. Counterexample: take $X = \{1, 2, 3\}$ and $R = \{(1, 2), (3, 2)\}$. Then $R^{-1} = \{(2, 1), (2, 3)\}$. Hence, $RoR^{-1} =$

$\{(2,2)\}$ and $R^{-1}oR = \{(1,1),(1,3),(3,1),(3,3)\}$.

(3) True. The proof was done in class.

(4) Straightforward.

(5) Easy.

Question: Let R be a relation on X. Prove or disprove:

(1) RoR^{-1} is transitive.

(2) RoR^{-1} is reflexive.

Question: Let R be the relation on $\mathbb{Z}$ defined by:

$$aRb \text{ iff } 3a+b \text{ is a multiple of} 4.$$

Is R an equivalence relation on $\mathbb{Z}$? If yes, then what is $[1]$. Also, find R^{-1}.

Solution: R is an equivalence relation, because

(1) R is reflexive, because $\forall a \in Z$, $3a+a=4a$ which is divisible by 4. Thus, $(a,a) \in \mathbb{R}$.

(2) R is symmetric, because if $(a,b) \in R$, then $3a+b=4m$, for some $m \in \mathbb{Z}$. Thus, $b = 4m-3a$. But now $3b+a = 3(4m-3a)+a = 4(3m-2a)$. Notice that $3m-2a$ is an integer. Thus, $3b+a$ is divisible by 4, which means that $(b,a) \in R$.

(3) R is transitive, because if (a,b) and (b,c) are in R, then $3a+b=4m$ for some $m \in \mathbb{Z}$ and $3b+c=4k$ for some $k \in \mathbb{Z}$.

Therefore, $3a + c = 4m - b + 4k - 3b = 4(m + k - b)$. Notice that $m + k - b$ is an integer. Thus, $3a + c$ is divisible by 4, which means $(a, c) \in R$.

$$\begin{aligned}
[1] &= \{b \in \mathbb{Z} \mid aR1\} \\
&= \{a \in \mathbb{Z} \mid 3a + 1 = 4k,\ k \in \mathbb{Z}\} \\
&= \{a \in \mathbb{Z} \mid a = \frac{4k - 1}{3},\ k \in \mathbb{Z}\} \\
&= \{a \in \mathbb{Z} \mid a = 4k - 3,\ k \in \mathbb{Z}\}
\end{aligned}$$

$R^{-1} = \{(a, b) \mid a + 3b \text{ is a multiple of } 4\}$.

Question: Define the following relation on R^2:

$$(a, b)R(c, d) \text{ iff } a^2 + b^2 = c^2 + d^2.$$

Is R an equivalence relation? If yes, then what is the equivalence class of (α, β) and what does it represent in the Cartesian plane? Also, find R^{-1}.

Solution: R is an equivalnce relation. This can be easily verified. Also, it is easy to see that $R^{-1} = R$. Why?

Now, $[(\alpha, \beta)] = \{(x, y) \mid x^2 + y^2 = \alpha^2 + \beta^2\}$. It is easy to see that $[(\alpha, \beta)]$ is a circle centered at the origin and with readius $\sqrt{\alpha^2 + \beta^2}$. Actually, the disjoint equivalence classes partition the Cartesian plane (i.e. R^2) into a set of circles centered at the origin.