Coordinates and Change of Basis

Notation: We denote the ordered basis $v_1,\ v_2, \cdots,\ v_n$, by $[v_1, v_2, \cdots, v_n]$ and the standard (natural) ordered basis by $S = [e_1, e_2, \cdots, e_n]$. Note that e_i is the $n-$column vector whose i^{th} entry is 1 and the remaining entries are zeros.

Definition: Let $T = [v_1, v_2, \cdots, v_n]$ be an ordered basis for a vector space V and let $v \in V$. Then v can be written uniquely as a linear combination of the vectors of T. I.e. there are constants $c_1, c_2, \cdots, c_n$ such that $v = \sum_{i=1}^{n} c_i v_i$. We call $(c_1, c_2, \cdots, c_n)$ the *coordinates* of v with respect to T. Note that the order of the vectors of the ordered basis is important. For example, the coordinates of v with respect to $T_2 = [v_2, v_1, v_3, \cdots, v_n]$ are $(c_2, c_1, c_3, \cdots, c_n)$ and the coordinates of v with respect to $T_3 = [v_3, v_1, v_2, \cdots, v_n]$ are $(c_3, c_1, c_2, \cdots, c_n)$.

Notation: Let V be a vector space and let $v \in V$ and T be an ordered basis for V. We denote the coordinates of v with respect to T by $[v]_T$.

Theorem: Let T_1 and T_2 be two ordered bases for a vector space V of dimension n. The matrix to change the coordinates of a vector $v \in V$ from basis T_1 to T_2 is called a *transition matrix*. I.e. if we call this transition matrix A, then $[v]_{T_2} = A[v]_{T_1}$ and $[v]_{T_1} = A^{-1}[v]_{T_2}$. Note that if the transition matrix from T_1 to T_2 is A, then the transition matrix from T_2 to T_1 is A^{-1}. Note also that A is invertible. Why?

Theorem: Let $S = [e_1, e_2, \cdots, e_n]$ be the standard basis for R^n and $T_1 = [v_1, v_2, \cdots, v_n]$ and $T_2 = [w_1, w_2, \cdots, w_n]$ be two other ordered bases for R^n, where all vectors involved are column vectors. Then

(1) The transition matrix from from T_1 to S is $A_1 = [v_1\ v_2\ \cdots\ v_n]$ (i.e. A_1 is the matrix whose columns are the column vectors $v_1,\ v_2, \cdots, v_n$).

(2) The transition matrix from S to T_1 is A^{-1}. Note that A is invertible because T_1 is a basis, and hence, it's linearly independent, which implies $\det(A_1) \neq 0$.

(3) Now let $A_2 = [w_1\ w_2\ \cdots\ w_n]$ be the transition matrix from T_2 to S. Then the transition matrix from T_1 to T_2 is $B = {A_2}^{-1}A_1$ and the transition matrix from T_2 to T_1 is $B^{-1} = {A_1}^{-1}A_2$.

(4) Thus, if $v \in V$, then $[v]_{T_2} = B[v]_{T_1}$ and $[v]_{T_1} = B^{-1}[v]_{T_2}$.

Remark: Everything we said about R^n applies to P_{n-1} (polynomials of degree $\leq n-1$). All you need to do here is to represent every polynomial by its coefficients.