Inverse and Its Properties

Definition: Let $A = (a_{ij})$ be an $n \times n$ matrix. The **inverse** of A, denoted A^{-1}, is the matrix B such that $AB = I_n$, where I_n is the $n \times n$ identity matrix.

Note that the inverse is defined only for square matrices. But not all square matrices have inverses. Note also that if B is the inverse of A, then B is the inverse of A. If a square matrix has inverse, we say it is **invertible** or **nonsingular**. If a square matrix has no inverse, we say it is singular or not invertible. An $n \times n$ matrix A is invertible if and only if $\det(A) \neq 0$. Although matrix multiplication is not commutative in general, if $AB = I$, then $BA = AB = I$.

The matrix inverse was first published by Cayley (from England) in 1853. Although matrices were studied (e.g. magic squares and latin squares) thousands of years ago, the term *matrix* was first introduced in 1848 by Sylvester from England. Some of the people who did significant work on matrices in the last few centuries are the German Leibniz (he worked on determinants and he is also one of the two founders of calculus) who developed the theory of determinants in 1693, the Japanese Takakazu in 1683, the Swiss Cramer who developed the theory of determinants more and introduced Cramer's rule in 1750, the German Gauss and Jordan who developed Gaussian elimination and Gauss-Jordan elimination in the ninteeth century (although there is an example on Gaussian elimination that was done about 200 BC), Cayley, Hamilton (Irish), Grassmann (German), Frobenius (German), Von Neumann (Hungarian), and Todd (Austria).

Applications of Inverses: Solving linear systems and many others.

Definition: Let $A = (a_{ij})$ be an $n \times n$ matrix. The **adjoint** of A denoted, adj A, is the $n \times n$ matrix whose ij^{th} entry is A_{ji} (the cofactor of a_{ji}). I.e.

$$\text{adj } A = (A_{ij})^T = (A_{ji}).$$

I.e. we obtain the adjoint of A be taking the transpose of the matrix of cofactors.

Definition: Let A be an invertible $n \times n$ matrix. Then

$$A^{-1} = \frac{1}{\det(A)} \text{adj} A.$$

Theorem: Let A and B be $n \times n$ matrices and let α be a nonzero number and k be a positive integer. Then

(1) If A is invertible, then so are A^{-1}, A^k, A^T, αA, and $(A^{-1})^{-1} = A$, $(A^k)^{-1} = (A^{-1})^k$, $(A^T)^{-1} = (A^{-1})^T$, $(\alpha A)^{-1} = \frac{1}{\alpha} A^{-1}$. In particular, $(-A)^{-1} = -(A^{-1})$.
(2) If A and B are invertible, then so is AB, and $(AB)^{-1} = B^{-1}A^{-1}$.
(3) In general, if A and B are invertible, $A + B$ is not necessarily invertible (e.g. take $A = I_n$ and $B = -A$), and even if $A + B$ is invertible, in general $(A + B)^{-1} \neq A^{-1} + B^{-1}$.
(4) $\det(A^{-1}) = \frac{1}{\det(A)}$.
(5) A is invertible iff $\det(A) \neq 0$. Or equivalently, A is singular iff $\det(A) = 0$.
(6) A is invertible iff the system $Ax = 0$ has only the trivial solution.
(7) A is invertible iff the system $Ax = b$ has a unique solution for every $n \times 1$ vector b.
(8) A is invertible iff it's row equivalent to I_n.
(9) Let e_i be the n-vector whose i^{th} entry is 1 and all other entries are zeros. The solution of $Ax = e_i$ gives column i of A^{-1}. Thus, finding A^{-1} is equivalent to solving the n linear systems of equations:

$$Ax = e_i, \ i = 1, \cdots, n.$$