Elementary Properties of Determinants

Definition: Let $A = (a_{ij})$ be an $n \times n$ matrix. The **cofactor** of entry a_{ij} of A, denoted A_{ij}, is $(-1)^{i+j} \det(M_{ij})$, where M_{ij} is the matrix obtained from A by deleting row i and column j.

Definition: Let $A = (a_{ij})$ be an $n \times n$ matrix. The **determinant** of A denoted, $\det(A)$ or $|A|$, is

$$\det(A) = \sum_{j=1}^{n} a_{ij} A_{ij}, \; i = 1, \cdots, n.$$

This is called the determinant/cofactor expansion across/along row i.

Also,

$$\det(A) = \sum_{i=1}^{n} a_{ij} A_{ij}, \; j = 1, \cdots, n.$$

This is called the determinant/cofactor expansion across/along column j.

Some of the people who did a significant work on determinants in the last 3 centuries are the Japanese Takakazu in 1683, the German Leibniz in 1693, the Swiss Cramer in 1750 (he used them in analytic geometry and solving linear systems), the French Cauchy in the ninteeth century who used them also in analytic geometry, and the Scotch Muir.

Applications of Determinants: Solving linear systems, determining if a matrix is singular, finding the inverse, finding eigenvalues, and many others.

Theorem: Let A and B be $n \times n$ matrices and let α be a number. Then

(1) $\det(A^T) = \det(A)$.

(2) $\det(A^m) = (\det(A))^m$, $\forall m \in \mathbb{Z}$.

(3) $\det(AB) = \det(A) \cdot \det(B)$.

(4) $\det(\alpha A) = \alpha^n \det(A)$.

(5) If A is nonsingular (invertible), then $\det(A^{-1}) = \frac{1}{\det(A)}$.

(6) If A is diagonal or lower-triangular or upper-triangular, then the determinant of A is equal to the product of its (main) diagonal elements.

(7) If A has a zero row/column, then $\det(A) = 0$.

(8) If two rows/columns of A are equal, then $\det(A) = 0$.

(9) Interchaing two rows of A results in multiplying the determinant by -1.

(10) Adding a multiple of a row of A to another of the forma $r_k + \alpha r_m \longrightarrow r_k$ has no effect on the determinant. But, $r_m + \alpha r_k \longrightarrow r_k$ does change the determinant (it multiplies the determinant by α).

(11) Multiplying a row/column of A by α results in multiplying the determinant by α.

(12) A is invertible (nonsingular) iff $\det(A) \neq 0$.

(13) $\det(A) = 0$ iff A has a zero eigenvalue.

(14) $Ax = b$ has a unique solution (which is $A^{-1}b$) iff $\det(A) \neq 0$.

(15) $Ax = 0$ has a unique solution (which is the trivial solution) iff $\det(A) \neq 0$.

(16) If A is a skew-symmetric matrix of odd order (i.e. n is odd), then $\det(A) = 0$.

(17) A (adj A) = (adj A) $A = \det(A)I_n$, where I_n is the $n \times n$ identity matrix.

(18) If A is row-equivalent to B, then A and B don't necessarily have the same determinant. Also, in general, $\det(A + B) \neq \det(A) + \det(B)$.

(19) If A is orthogonal, then $\det(A) = 1$ or $\det(A) = -1$.

(20) If A is unitary, then $|\det(A)| = 1$. (Note that $\det(A)$ can be complex.)

(21) If A is orthogonal, then $\det(A) = 1$ or $\det(A) = -1$.