Questions

(1) Determine whether the function $f:\ N \longrightarrow \mathbb{Z}$ defined by $f(x) = f(x-1) + f(x-2)$, $f(1) = 2$, $f(2) = -1$, is one-to-one.

(2) Find the inverse of the function $f:\ \mathbb{R}^+ \longrightarrow (1, e)$, defined by $f(x) = e^{\frac{x}{x+3}}$.

(3) (a) Let $p \in \mathbb{N}$, $p \geq 624$, and let $w \in \mathbb{R}$, $|w| \neq 1$. Use mathematical induction to prove that:

$$w^p + w^{p+1} + ... + w^{p+n-1} + 2 = 2 + \tfrac{w^p - w^{p+n}}{1-w},\ \forall n \in \mathbb{N}..$$

(b) Find the following sum. Do not use calculators.

$$2^{5553} + 8 + 16 + 32 + 64 + 128 + ... + 2^{5552}.$$

(4) Prove or disprove: If $f : A \longrightarrow B$ is a bijective function, then $f^{-1} : B \longrightarrow A$ is defined and it is also bijective.

(5) Show using the defintion of one-to-one that the function $f : \mathbb{N} \longrightarrow [e^{\frac{1}{4}}, e)$ defined by $f(x) = e^{\frac{x}{x+3}}$ is one-to-one.

(6) Prove or disprove: If A and B are subsets of a universal set X such that $A \subseteq B$, then $A^c \subseteq B^c$.

(7) Prove or disprove: The set $A = \{m | m = 1 + 10n, n \in \mathbb{N}\}$ is a proper subset of the set $B = \{m | m = 1 + 5n, n \in \mathbb{N}\}$.

(8) Prove or disprove: If a and b are irrational numbers, then $a(b + 1) + b$ is an irrational number.

(9) Let A be any set and let R be a relation on R.

(a) Find sufficient and necessary conditions for R to be symmetric and antisymmetric.

(b) Find sufficient and necessary conditions for R to be symmetric and not antisymmetric.

(c) Find sufficient and necessary conditions for R to be not symmetric and antisymmetric.

(d) Find sufficient and necessary conditions for R to be not symmetric and not antisymmetric.

(e) Find sufficient and necessary conditions for R to be equivalence and antisymmetric.

(f) Find sufficient and necessary conditions for R to be equivalence and not antisymmetric.

(g) If R is an equivalence relation, find a necessary and sufficient condition for the equivalence classes of R to consist of one element (each).

(10) Let A be the set of all first degree polynomials with integer coefficients. Define the relation R on A to be

$$\{(f,g)|f \text{ and } g \text{ have the same slope}\}.$$

Is R an equivalence relation? If yes, then what are the equivalence classes of $f(x) = -999$ and of $g(x) = -999x + 777$.

(11) Let A be the set of all continuous functions. Define the relation R on A to be

$$\{(f,g)|f(0) = g(0) + 3k,\ \ k \in \mathbb{Z}\}.$$

Is R an equivalence relation on A? If yes, then what are the equivalence classes of $f(x) = -999$ and of $g(x) = -999x^{777}$.

(12) Define the following relation R on $\mathbb{R}^2$:

$$(a,b) \sim (c,d) \text{ iff } |3a - 2b| = |3c - 2d|.$$

Is R an equivalence relation on $\mathbb{R}^2$? If yes, then give five elements in the equivalence class of (1,2). Does $(\frac{1}{3}, \frac{5}{9}) \in \overline{(1,2)}$? Is $\overline{(1,2)} = \overline{(-1,2)}$? Explain.

(13) Let R be the relation on $\mathbb{R}^2$ defined by:

$$(x,y)R(c,d) \text{ iff } x + 2y = c + 2d.$$

Find $[(1,3)]$. What does $[(1,2)]$ represent in the Cartesian plane?

(14) Let R be the relation on $\mathbb{Z}$ defined by

$$a\ R\ b \text{ if and only if } a = |b|^3.$$

Is R antisymmetric? Is is transitive? If the answer to any of them is yes, prove that. If the answer is not, then write down a counter example. Also, find R^{-1}.

(15) Let R be the relation on $\mathbb{Z}$ defined by

$$a\ R\ b \text{ if and only if } 3 - b \leq a - 2b < 1 + b.$$

Give a counter example to show that R is not transitive. Is R symmetric? If yes, prove it. If not write doen a counterexample. Think (but do not hand in) about how to find R^{-1}.

(16) Let $A = \{(x, y) \in \mathbb{R}^2 \mid x^2+y^2 \leq 1\}$, $B = \{(x, y) \in \mathbb{R}^2 \mid |x| \leq 1 \text{ and } |y| \leq 1\}$. Let $M = \mathbb{Z} \cup \{\frac{1}{2}\}$ and $L = \{x \mid x = \frac{1}{2} + q, q \in \mathbb{Z}\}$. Finally, let $C = A \cap (M \times M)$, $D = B \cap (M \times M)$, $E = A \cap (L \times L)$, $F = B \cap (L \times L)$. Find

(a) $D - C$.

(b) $F \cap D$.

(17) Prove the following formula by mathematical induction:

$$(2 + 4 + 6 + 8 + ... + 2n)^3 = n^3(n + 1)^3, \forall n \in \mathbb{N}.$$

Hint: It suffices to prove that

$$2 + 4 + 6 + ... + 2n = n(n + 1), \forall n \in \mathbb{N}.$$

(18) Let $X = \{e, f, g\}$. Give an example of a binary relation on X which is symmetric and antisymmetric at the same time.

(19) Let R be a relation on a set X. If $(a, b) \in RoR^{-1}$, prove that (a, a) and (b, b) are both in RoR^{-1}. Give full explanation and write down every step.

(20) Let U be the unit disc in $\mathbb{R}^2$, $X = \{x + \frac{1}{5} \mid x \in Z\}$ and let $Y = \mathbb{Z} \cup \{\frac{1}{5}\}$. Also, let $G = \{(x, y) \in X \times X| \; |x| \leq \frac{4}{7}\}$, $A = U \cap (Y \times Y)$, $B = U \cap G$, and $C = U \cap (\mathbb{Z} \times \mathbb{Z})$. Find

(a) $A \triangle B$.

(b) $A \triangle C$.

(c) $B \triangle C$.

(d) $A \cup B$.

(e) $A \cup C$.

(f) $B \cup C$.

(21) Find $\mathcal{P}(\mathcal{P}(\{\phi, \{0, \{1\}, a\}\}))$.

(22) Define the following relation on R^2:

$$(a, b)R(c, d) \text{ iff } a^2 + b^2 = c^2 + d^2.$$

Find $[(\alpha, \beta)]$. What does $[(\alpha, \beta)]$ represent in the Cartesian plane? Also, find R^{-1}.

(23) Let $\{u_n\}_{n=1}^{\infty}$ be defined by

$$u_n = \tfrac{2}{3} - (\tfrac{1}{6})(\tfrac{1}{4^{n-2}}),\ \forall n \in \mathbb{N}.$$

Let $A = (-5, \frac{11}{3}] \cap [\frac{7}{4}, 5) \cap \mathbb{Z}^+$ and let $S = A - \{-4, 7\}$. Find

$$\Sigma_{i \in S}\ u_i$$

.

(24) Let

$$A = \{(-1,2),(4,5),(0,0),(6,-5),(5,1),(4,3)\}$$
$$B = \{b \mid b = k^2 \text{ for some } k \in \mathbb{Z} \text{ and } (a,b) \in A \text{ for some } a\}.$$
$$C = \{x - 4 \mid x \in Z \text{ and } \tfrac{-157}{7x^2-35x+42} \geq 0\}.$$

Find $(C \cup B) \cap \{\{-3,1,2\},\{1\},0,5,4,\{2\},\{0\},-2,-1,\phi\}$.

(25) Let $A = \mathbb{Z} \times \mathbb{Z}$, and let $B = \{(x,y) \in \mathbb{R}^2 \mid x^2 + y^2 \leq 1\}$. Find $A \cap B$.

(26) Let $\{u_n\}_{n=1}^{\infty}$ be defined by

$$u_n = \tfrac{2}{3} - (\tfrac{1}{6})(\tfrac{1}{4^{n-2}}),\ \forall n \in \mathbb{N}.$$

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$$B = \{b \mid b = k^2 \text{ for some } k \in \mathbb{Z} \text{ and } (a,b) \in A \text{ for some } a\}.$$
$$C = \{x - 4 \mid x \in Z \text{ and } \tfrac{x^2-5x+6}{-1576} \geq 0\}.$$

Find $(C \cup B) \cap \{\{-3,1,2\},\{1\},0,5,4,\{2\},\{0\},-2,-1,\phi\}$.

(28) Let R be the relation on $\mathbb{N}$ defined by: $a\ R\ b$ iff $\frac{a}{b}$ is a natural number. Prove by a counter example that R is not symmetric.

(29) Find -125,617 (mod 315).

(30) Find $0 \leq r \leq 10$ such that $317 \equiv r \pmod{11}$

(31) Find $0 \leq r \leq 10$ such that $-317 \equiv r \pmod{11}$

(32) To which equivalence class does 317 belong? And to wich equivalence class (mod 11) does -317 belong?

(33) Find $0 \leq w, z \leq 10$ such that $\overline{317} = \overline{w}$ and $\overline{-317} = \overline{z}$ (of course, here everything is (mod 11).)

(34) Find all values of x (if any), $0 \leq x < 6$, which solve the following conqruence equation:

$$4x^2 + 3x + 2 \equiv 0 \pmod 6$$

(35) Solve the following congruence system:

$2x + 3y \equiv 3 \pmod 7$

$3x + 5y \equiv 4 \pmod 7$

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(iii) Find $0 \leq r \leq 10$ such that $-317 \equiv r \pmod{11}$

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(38) Solve the following congruence system:

$2x + 3y \equiv 3 \pmod 7$

$3x + 5y \equiv 4 \pmod 7$

(39) Solve the following equation:

$$x \equiv 2x(3x + 2) + 3 \pmod 7$$

(40) Let a and b be nonzero integers and let $gcd(a, b) = g$. Also, let m and n be integers satisfying:

$$g = ma + nb.$$

(a) Let M_1 and N_1 be integers satisfying:

$$gcd(a, -b) = M_1 a + N_1(-b).$$

Express M_1 in terms of m and N_1 in terms of n.

(b) Express the least common multiple of $-a$ and $-b$ in terms of g, a, and b.