Discrete Mathematics/Structures

Examples on Many Topics

Question 1: Let $A = \mathbb{Z} \times \mathbb{Z}$, and let $B = \{(x, y) \in \mathbb{R}^2 \mid x^2 + y^2 \leq 1\}$. Find $A \cap B$.

Solution: $\{(-1, 0), (0, -1), (0, 0), (1, 0), (0, 1)\}$.

Question 2: Let $\{u_n\}_{n=1}^{\infty}$ be defined by

$$u_n = \frac{2}{3} - (\frac{1}{6})(\frac{1}{4^{n-2}}), \forall n \in \mathbb{N}.$$

Let $A = (-5, \frac{11}{3}] \cap [\frac{7}{4}, 5) \cap \mathbb{Z}^+$ and let $S = A - \{-4, 7\}$. Find

$$\Sigma_{i \in S}\ u_i$$

.

Solution: $A = \{2, 3\}$. $S = \{2, 3\}$. Therefore, $\sum_{i \in S} u_i = u_2 + u_3 = \frac{9}{8}$.

Question 3: Let

$$A = \{(-1, 2), (4, 5), (0, 0), (6, -5), (5, 1), (4, 3)\}$$

$$B = \{b \mid b = k^2 \text{ for some } k \in \mathbb{Z} \text{ and } (a, b) \in A \text{ for some } a\}.$$

$$C = \{x - 4 \mid x \in Z \text{ and } \frac{x^2 - 5x + 6}{-1576} \geq 0\}.$$

Find $(C \cup B) \cap \{\{-3, 1, 2\}, \{1\}, 0, 5, 4, \{2\}, \{0\}, -2, -1, \phi\}$.

Solution: $B = \{0, 1\}$. $C = \{-2, -1\}$. Therefore,

$$(C \cup B) \cap \{\{-3, 1, 2\}, \{1\}, 0, 5, 4, \{2\}, \{0\}, -2, -1, \phi\} = \{0, -1, -2\}.$$

Question 4: Prove by mathematical induction:

$$(1+2+3+...+n)^2 = 1+2^3+...+n^3, \forall n \in \mathbb{N}.$$

Solution: First, prove that $1+2+3+...+n = \frac{n(n+1)}{2}$, $\forall n \in \mathbb{N}$. We did that on Quiz 1. Or you can depend on the handout about arithmetic and geometric sequences. It is easy to see that the previously mentioned sequence is an arithmetic sequence with $a = d = 1$. Now depend on the formula for S_n (the sum of the first n terms). Then, replace $(1+2+3+...+n)^2$ by $\frac{n^2(n+1)^2}{4}$ and prove the new result by induction. (We did that as an example in class.)

==

Question 1: How many six-digit decimal numbers with no repetitions start with 57 or 75?

Solution: $2.P(8,4) = 2.8.7.6.5$.

Question 2: If repetitions are not allowed, what is the number of seven-digit odd decimal numbers?

Solution: 8.8.7.6.5.4.5.

Question 3: How many eight-digit decimal strings read the same from either side?

Solution: 10^4. Notice that if it is numbers instead of strings, then the answer would be 9.10^3.

Question 4: Let $f : \mathbb{N} \times \mathbb{N} \longrightarrow \mathbb{Z}$ be defined by

$$f(m,n) = m^2 - 2n + 5.$$

Prove by a counterexample that f is not one-to-one.

Solution: $f(4,8) = f(2,2)$.

Question 5: In how many ways can 5 red (*identical*) balls and 7 blue (*identical*) balls be distributed into 20 distinct boxes with at most one ball to a box?

Solution: $C(20,5).C(15,7)$ or $C(20,7).C(13,5)$. Notice if there is no limit on the numbers of balls in a box, then the answer would be $C(24,5).C(26,7)$.

Question 6 Find the coefficient of x^{13} in the binomial expansion of $(3+2x)^{20}$.

Solution: $(3+2x)^{20} = \sum_{k=0}^{20} C(20,k)3^k(2x)^{20-k}$. We want $20-k=13$. This implies $k=7$. Hence, the coefficient of x^{13} is $C(20,7).3^7.2^{13}$.

==

Question 1: Prove the following by mathematical induction:

(a) $(6)(7^n) - (2)(3^n)$ is divisible by 4, for every $n \in \mathbb{N}$.

Solution:

(1) *Basic Step:*

$(6)(7) - (2)(3) = 36$ is divisible by 4. Thus, the statement is true for $n = 1$.

(2) *Inductive Step:*

Assume that the statement is true for $n = k$, where $k \in \mathbb{N}$. In other words, assume that

$$(6)(7^k) - (2)(3^k) = 4m, \text{ for some } m \in \mathbb{Z}.$$

Notice that the above equation implies that $(6)(7^k) = 4m + (2)(3^k)$. Now depend on that to prove that the statement is true for $n = k+1$. In other words, prove that

$$(6)(7^{k+1}) - (2)(3^{k+1}) = 4q, \text{ for some } q \in \mathbb{Z}.$$

But,

$$\begin{aligned}(6)(7^{k+1}) - (2)(3^{k+1}) &= (7)[(6)(7^k)] - (2)(3)(3^k)\\ &= 7[4m + (2)(3^k)] - (6)(3^k)\\ &= 28m + (14)(3^k) - (6)(3^k)\\ &= 28m + (14-6)(3^k)\\ &= 28m + (8)(3^k)\\ &= 4[7m + (2)(3^k)].\end{aligned}$$

Notice that $7m + (2)(3^k) \in \mathbb{Z}$. Hence, $(6)(7^{k+1}) - (2)(3^{k+1})$ is divisible by 4. Hence, the statement is true for $n = k + 1$.

Therefore, by mathematical induction, $(6)(7^n) - (2)(3^n)$ is divisible by 4, for every $n \in \mathbb{N}$.

(b) $(\frac{3}{2})^n \geq 1 + \frac{n}{2}$, for all $n \in \mathbb{N}$.

Solution:

(1) *Basic Step:*

$(\frac{3}{2})^1 \geq 1 + \frac{1}{2}$. Thus, the statement is true for $n = 1$.

(2) *Inductive Step:*

Assume that the statement is true for $n = k$, where $k \in \mathbb{N}$. In other words, assume that

$$(\tfrac{3}{2})^k \geq 1 + \tfrac{k}{2}$$

Now depend on that to prove that the statement is true for $n = k+1$. In other words, prove that

$$(\tfrac{3}{2})^{k+1} \geq 1 + \tfrac{k+1}{2} = \tfrac{3}{2} + \tfrac{k}{2}.$$

But,

$$\begin{aligned}(\frac{3}{2})^{k+1} &= (\frac{3}{2})(\frac{3}{2})^{k}\\ &\geq (\frac{3}{2})(1+\frac{k}{2})\\ &= \frac{3}{2}+\frac{3}{4}k\\ &\geq \frac{3}{2}+\frac{k}{2}.\end{aligned}$$

Thus,

$$(\tfrac{3}{2})^{k+1} \geq 1+\tfrac{k+1}{2} = \tfrac{3}{2}+\tfrac{k}{2}.$$

Hence, the statement is true for $n = k+1$.

Therefore, by mathematical induction, $(\frac{3}{2})^n \geq 1+\frac{n}{2}$, for all $n \in \mathbb{N}$.

Question 2: Prove or disprove:

(a) $(p \longrightarrow q) \longrightarrow r \equiv p \longrightarrow (q \longrightarrow r)$.

(b) $p \wedge (q \vee r) \equiv (p \wedge q) \vee (p \wedge r)$.

Question 3: Prove or disprove:

(a) $x^4 - 4x^2 + 4$ is nonnegative for every real number x.

Solution:

The statement is true, because $x^4 - 4x^2 + 4 = (x^2-2)^2 \geq 0, \forall x \in \mathbb{R}$. Notice that the function $f(t) = t^2$ is nonnegative for every real number.

(b) $(n^2 - 5n + 6)^3 + (1 + (-1)^n)(2n+1)$ is an even integer for every natural number n.

Solution:

The statement is true. First, we have proved in class (you need to do it here) that $n^2 - 5n + 6$ is even for all natural numbers. Thus, $n^2 - 5n + 6 = 2q$, for some $q \in \mathbb{Z}$, and for all $n \in \mathbb{N}$. Now we have the following two cases:

Case 1: n is odd. In this case,

$$(n^2 - 5n + 6)^3 + (1 + (-1)^n)(2n + 1) = (2q)^3 + (0)(2n + 1) = 8q^3 = 2(4q^3).$$

Since $4q^3$ is an integer, it follows that $(n^2 - 5n + 6)^3 + (1 + (-1)^n)(2n + 1)$ is even in this case.

Case 2: n is even. In this case,

$$(n^2 - 5n + 6)^3 + (1 + (-1)^n)(2n + 1) = (2q)^3 + (2)(2n + 1) = 2(4q^3 + 2n + 1).$$

Since $4q^3 + 2n + 1$ is an integer, it follows that $(n^2 - 5n + 6)^3 + (1 + (-1)^n)(2n + 1)$ is even in this case also.

Since there are no other possibilities, the statement is true for all $n \in \mathbb{N}$.

(c) The sum of every two different prime numbers is an even integer.

Solution:

The statement is false. *Counterexample:* Take the two prime numbers 2 and 3. The sum of these two prime numbers is 5, which is not even.

(d) If a and b are irrational numbers, then a(b+1)+b is an irrational number.

Solution:

The statement is false. *Counterexample:* Take $a = \sqrt{2}$ and take $b = -a$. Then

$$a(b + 1) + b = ab + a + b = ab = -(a^2) = -2$$

Thus, a and b are both irrational, but $a(b+1)+b$ is rational.

(e) The system

$$3x - y = 1$$

$$-6x + 2y = 7$$

has no real solution.

Solution: True. The proof is by contradiction. Assume the system has a real solution and try to slove it, you'll get something like $0 = 7$ which is absurd.

Question 4: Decide whether the following statements are true or false. Explain why.

(a) 8 is a prime number and 8 is an even integer $\longrightarrow$ either $\sqrt{5} = 1$ or $-2 > 0$.

Solution:

The statement is true. Notice that we have

$$(F \wedge T) \longrightarrow (F \vee F),$$

which simplifies to

$$F \longrightarrow F.$$

(b) If the only y-intercept of $x^2 + 4y^2 - 1 = 0$ is 1, then the only x-intercept of $e^x - 4y - 3 = 0$ is $ln(3)$.

Solution:

The statement is true. Notice that we have

$$F \longrightarrow T.$$

==

Question 1:

(a) $D - C = \{(1,1),(1,-1),(-1,\frac{1}{2}),(1,\frac{1}{2}),(-1,1),(-1,-1),(\frac{1}{2},1),(\frac{1}{2},-1)\}$.

(b) $F \cap D = \{(\frac{1}{2},\frac{1}{2})$.

Question 2:

(a) The function $f(x) = 2x$ if $x \in \mathbb{Z}^+$ and $f(x) = 1 - 2x$ if $x \in Z^- \cup \{0\}$ is a bijective function from $\mathbb{Z}$ onto $\mathbb{N}$. You have to prove that it is bijective or you can depend on the following

- The function $g: \mathbb{N} \longrightarrow \mathbb{Z}$, defined by $g(x) = x/2$ if x is even, and $g(x) = (1-x)/2$ if x is odd, is bijective.
- If $f: A \longrightarrow B$ is bijective, then so is $f^{-1}: B \longrightarrow A$.

(b) We want to prove that the function $f: \mathbb{N} \longrightarrow [e^{\frac{1}{4}}, e)$ defined by $f(x) = e^{\frac{x}{x+3}}$ is one-to-one.

Let x_1 and x_2 be in $\mathbb{N}$ and assume that $f(x_1) = f(x_2)$. Then $e^{\frac{x_1}{x_1+3}} = e^{\frac{x_2}{x_2+3}}$. Now take the natural logarithm of both sides to get that $\frac{x_1}{x_1+3} = \frac{x_2}{x_2+3}$. By cross multiplying, we get $x_1(x_2+3) = x_2(x_1+3)$. Thus, $x_1x_2 + 3x_1 = x_2x_1 + 3x_2$. Hence, $x_1 = x_2$.

Question 3:

(a) Let R be the relation on $\mathbb{Z}$ defined by

$$a \; R \; b \text{ if and only if } a = |b|^3.$$

Is R antisymmetric? Is is transitive? If the answer to any of them is yes, prove that. If the answer is not, then write down a counter example. Also, find R^{-1}.

R is antisymmetric. To prove that use the following definition which I gave in class:

$$R \text{ is antisymmetric iff whenever } (a,b) \text{ and } (b,a) \text{ are in } R, \text{ then } a = b.$$

So, assume that (a,b) and (b,a) are in R for some a, b in $\mathbb{Z}$. Now $(a,b) \in R$ implies that $a = |b|^3$ and $(b,a) \in R$ implies that $b = |a|^3$. Thus, $a = |a|^9$. But, $|a|^9$ is nonegative. Hence, a is nonnegative. Thus, we can get rid of the absolute value, to get $a = a^9$. Therefore, either $a = 0$ or $a = 1$. If $a = 0$, then $b = 0$ and if $a = 1$, then $b = 1$.

R is not transitive, because, for example, $(512, 8)$ and $(8, 2)$ are both in R, but $(512, 2)$ is not. Notice that $512 \neq 2^3$.

Notice that $R^{-1} = \{(a,b) \mid b = |a|^3\}$.

(b) The relation R on $\mathbb{Z}$ defined by

$$a \; R \; b \text{ if and only if } 3 - b \leq a - 2b < 1 + b.$$

is not transitive, because, for example, $(10, 5)$ and $(5, 2)$ are both in R, but $(10, 2)$ is not.

R is not symmetric, because, for example, $(10, 5) \in R$, but $(5, 10)$ is not.

(c) For the relation R on $\mathbb{Z}$ defined by

$$a \; R \; b \text{ if and only if } a - b \text{ is divisible by } 5.$$

$$\begin{aligned}
[17] &= \{a \in \mathbb{Z} \mid aR17\} \\
&= \{a \in \mathbb{Z} \mid a - 17 = 5k, \; k \in \mathbb{Z}\} \\
&= \{a \in \mathbb{Z} \mid a = 17 + 5k, \; k \in \mathbb{Z}\} \\
&= \{a \in \mathbb{Z} \mid a = 2 + 5m, \; m \in \mathbb{Z}\}.
\end{aligned}$$

Question 5: Let $\{a_n\}_{n=1}^{\infty}$ be defined as follows:

$$a_n = 1 + (-1)^n \text{ if } n \text{ is odd and } a_n = (n-1) + n \text{ if } n \text{ is even.}$$

Let $S_n = \sum_{k=1}^{n} a_k$.

(a) The general formula for S_n, $n \in \mathbb{N}$, is:

$S_n = \frac{n(n+1)}{2}$ if n is even and $S_n = \frac{n(n-1)}{2}$ if n is odd.

==

Question 1:

In how many ways can you order 7 math books, 9 computer science books, and 6 physics books on a shelf if

(a) all math, computer science, and physics books are distinct?

(b) all math, computer science, and physics books are distinct, and the math books are to be together?

(c) all math books are identical and all computer science and physics books are distinct?

(d) only 4 of the math books are identical and all computer science and physics books are distinct?

(e) Repeat part (b) for a circular shelf.

Question 2:

(a) In how many ways can you distribute 17 distinct balls into 25 distinct boxes at most one ball to a box?

(b) In how many ways can you distribute 17 distinct balls into 25 distinct boxes? (Here there is no limit on the number of balls in each box.)

(c) In how many ways can you distribute 17 red (identical) balls into 25 distinct boxes at most one ball to a box?

(d) In how many ways can you distribute 17 red (identical) balls into 25 distinct boxes? (Here there is no limit on the number of balls in each box.)

(e) In how many ways can you distribute 10 red (identical) balls and 7 blue (identical) balls into 25 distinct boxes at most one ball to a box?

(f) In how many ways can you distribute 10 distinct red balls and 7 blue (identical) balls into 25 distinct boxes at most one ball to a box?

(g) In how many ways can you distribute 10 red (identical) balls and 7 blue (identical) balls into 25 distinct boxes? (Here there is no limit on the number of balls in each box.)

(h) In how many ways can you distribute 10 distinct red balls and 7 blue (identical) balls into 25 distinct boxes? (Here there is no limit on the number of balls in each box.)

(i) A bag contains 10 red distinct balls and 7 blue distinct balls. In how many ways, can you draw 5 red and 4 blue?

Question 3:

(a) How many permutations of the letters $PRODUCTIVE$ contain a permutation of PRO?

(b) What is the number of permutations of the word $PRODUCT$ if no letter is to keep its original position?

Question 4:

(a) Find the middle term in the binomial expansion of $(3y - 2x)^{34}$. What is the coefficient of $x^{11}y^{23}$ in the previous expansion?

(b) Find the coefficient of x^{20} and the coefficient of x^{21} in the binomial expansion of $(4x^2 - \frac{1}{x})^{40}$.

Question 5:

(a) 100 points are to be distributed on 9 questions. If each question has to be out of at least 4. In how many ways can you distribute the points?

(b) What is the number of integer solutions of

$$x + y + z = 73,\ x \geq 0,\ y \geq 3,\ z > 7?$$

(c) A 6-member committee has to be chosen from a group of 9 mathematicians and 13 computer scientists. How many committees can be formed if the committee has to include two mathematicians only? And how many committees can be formed if the committee has to include at least two mathematicians?

Question 6:

(a) Find the number of odd 6-digit octal numbers if repetitions are not allowed?

(b) Find the number of 6-digit octal numberts that read the same from either side.

(c) Find the number of 7-digit octal strings that read the same from either side.

(d) If repetitions are not allowed, what is the number of 6-digit octal numbers that contain a 2 and a 5?

(e) If repetitions are not allowed, what is the number of 6-digit octal numbers that contain (*as a substring*) either 25 and 52?

(f) If repetitions are not allowed, what is the number of 6-digit octal numbers that start (from the left) with a 5 and end (from the right) with a 2 and which do not contain a 4?

(g) How many 6-digit octal numbers contain a 2 and do not contain a 5?

(h) If repetitions are not allowed, what is the number of 6-digit octal numbers that do not contain (*as a substring*) neither 25 nor 52?

==

Solution of the Last Group

Question 1:

(a) 22!. (b) $16! \cdot 7!$ (c) $\frac{22!}{7!}$

(d) $\frac{22!}{4!}$ (e) $15! \cdot 7!$.

Question 2:

(a) $P(25, 17)$ (b) 25^{17} (c) $C(25, 17)$

(d) $C(25 + 17 - 1, 17)$ (e) $C(25, 10) \cdot C(15, 7)$ (f) $P(25, 10) \cdot C(15, 7)$

(g) $C(25 + 10 - 1, 10) \cdot C(25 + 7 - 1, 7)$ (h) $25^{10} \cdot C(25 + 7 - 1, 7)$.

Question 3:

(a) $8! \cdot 3!$ (b) D_7.

Question 4:

(a) $C(34, 17) \cdot 3^{17} \cdot (-2)^{17} y^{17} x^{17}$, $C(34, 11) \cdot (-2)^{11} \cdot 3^{23}$

(b) $C(40, 20) \cdot 4^{20}$, 0.

Question 5:

(a) $C(72,64)$ (b) $C(64,62)$ $C(9,2) \cdot C(13,4)$.

Question 6:

(a) $6 \cdot 6 \cdot 5 \cdot 4 \cdot 3 \cdot 4$ (b) $7 \cdot 8^2$ (c) 8^4.

(d) $6 \cdot 5 \cdot P(6,4) - 5 \cdot 4 \cdot P(5,3)$ (e) $2 \cdot 5 \cdot P(6,4) - 2 \cdot 4 \cdot P(5,3)$

(f) $5 \cdot 4 \cdot 3 \cdot 2$ (g) $6 \cdot P(6,5) - 5 \cdot P(5,4)$ (h) $7 \cdot 7 \cdot 6 \cdot 5 \cdot 4 \cdot 3 - [2 \cdot 5 \cdot P(6,4) - 2 \cdot 4 \cdot P(5,3)]$.