CSIT 241 - Fall 02 - Exercises on Binary Relations

Remark: We will explain matrix representation of binary relations, digraph representation of binary relations, composition of binary relations, inverses of binary relations, Hasse diagrams of partial orders, and minimal, minimum, maximal, and maximum elements of partial orders. That means the following exercises do not cover all material about binary relations.

Read the definitions of asymmetric relations and circular relations.

The solutions I'll present below are brief. I'll skip some steps, but on quizzes/exams/homework/work you'll have to explain every step and write down everything.

(1) Decide whether the following binary relations on the indicated sets are reflexive, symmetric, antisymmetric, transitive, equivalence, partial orders. If the relation is an equivalence relation, find the equivalence classes.

1. $A = \mathbb{Z} - \{0\}$, $\mathcal{R} = \{(x, y) \in \mathbb{Z} \times \mathbb{Z} \mid x|y\}$.

   Solution:

   Reflexive, because every nonzero integer divides itself.

   Not symmetric, because, for example, 2 divides 4, but 4 does not divide 2.

   Not an equivalence relation, because not symmetric.

   Transitive, because if a, b, and c, are nonzero integers, and if $a|b$ and $b|c$, then $a|c$. This is a fact we talked about when we covered number theory.

   Not antisymmetric because, for example, $2|(-2)$ and $(-2)|2$, and $2 \neq -2$.

   Not a partial order because not antisymmetric.

2. $A = \mathbb{N}$, $\mathcal{R} = \{(x, y) \in \mathbb{N} \times \mathbb{N} \mid x|y\}$.

   Solution: This relation is reflexive, not symmetric, transitive, and not an equivalence relation. The proof is exactly the same as the previous exercise. But, this relation is antisymmetric, and hence, it's a partial order. It's antisymmetric, because if $(x, y) \in \mathcal{R}$ and if $(y, x) \in \mathcal{R}$, then $x = my$ and $y = nx$ for some itegers n and m. But, since y and x are positive, then m and n must be positive. Now notice that $x = mnx$. Thus, $mn = 1$. Since, m and n are both positive, it must be that $m = n = 1$. hence, $y = x$.

3. $A = \mathbb{Z}$, $\mathcal{R} = \{(x, y) \in \mathbb{Z} \times \mathbb{Z} \mid \frac{x}{y} \in \mathbb{Z}\}$.

4. $A = \{3, 4, 5, 6, 7\}$, $\mathcal{R} = \{(x, y) \in A \times A \mid x|y\}$.

5. $A = \mathbb{N}$, $\mathcal{R} = \{(x, y) \in \mathbb{N} \times \mathbb{N} \mid gcd(x, y) = 1\}$.

   Solution: Done in class.

6. A= set of all lines in the plane. $\mathcal{R} = \{(L_1, L_2) \mid L_1 || L_2\}$.

   Solution: Done in class.

7. Let S be a nonempty set. $A = \mathcal{P}(A)$. $\mathcal{R} = \{(C, D) \in A \times A \mid C \subseteq D\}$.

   Solution: Done in class.

8. $A = \mathbb{Z}$, $\mathcal{R} = \{(x, y) \in \mathbb{Z} \times \mathbb{Z} \mid 2|(x + y)\}$.

   Solution: Done in class.

9. $A = \mathbb{Z}$, $\mathcal{R} = \{(x, y) \in \mathbb{Z} \times \mathbb{Z} \mid x - y \in 2\mathbb{Z}\}$.

10. $A = \mathbb{Z} - \{0\}$, $\mathcal{R} = \{(x, y) \in A \times A \mid xy > 0)\}$.

    Solution: Done in class.

11. $A = \mathbb{Z} - \{0\}$. $(x, y) \in R$ iff $xy < 0$.

12. $A = \mathbb{R}$, $\mathcal{R} = \{(x, y) \in \mathbb{R} \times \mathbb{R} \mid x \le y\}$.

    Solution: Done in class.

13. $A = \mathbb{Z}$, $\mathcal{T} = \{(x, y) \in \mathbb{Z} \times \mathbb{Z} \mid x^2 = y^2\}$.

    Solution: Done in class.

14. $A = \mathbb{R}^2$, $\mathcal{R} = \{((a, b), (c, d)) \in \mathbb{R}^2 \times \mathbb{R}^2 \mid a^2 + b^2 = c^2 + d^2\}$.

    Solution: Done in class.

15. $A = \mathbb{Z}$, $\mathcal{R} = \{(x, y) \in \mathbb{Z} \times \mathbb{Z} \mid |x - y| < 3\}$.

    Solution:

    Reflexive, because if $x \in \mathbb{Z}$, then $|x - x| = 0 < 3$.

    Symmetric, because if $(x, y) \in R$, then $|x - y| < 3$. hence, $|y - x| < 3$, because $|y - x| = |x - y|$. Therefore, $(y, x) \in R$.

    Not transitive, because, for example, $(3, 5) \in R$ and $(5, 7) \in R$, and $(3, 7) \notin R$.

    Not an equivalence relation, because not transitive.

    Not a partial order, because not transitive.

    Not antisymmetric, because, for example, $(1, 2) \in R$ and $(2, 1) \in R$, and $1 \neq 2$.

16. $A = \mathbb{Z}$, $\mathcal{R} = \{(x, y) \in \mathbb{Z} \times \mathbb{Z} \mid x - y > 3\}$.

17. $A = \mathbb{Z}$, $\mathcal{R} = \{(x, y) \in \mathbb{Z} \times \mathbb{Z} \mid 3|(x^2 - y^2)\}$.

18. $A = \mathbb{Z}$, $\mathcal{R} = \{(x, y) \in \mathbb{Z} \times \mathbb{Z} \mid 3|(x - y)\}$.

Solution: A similar example is online (see the binary relations' material posted on my website). Here is the solution:

Reflexive, because if $x \in \mathbb{Z}$, then $x - x = 0 \cdot 3$. Hence, $x - x$ is a multiple of 3. Therefore, $(x, x) \in R$.

Symmetric, because if $(x, y) \in R$, then $x - y = 3m$, for some integer m. Thus, $y - x = -3m$. Hence, $(y, x) \in R$.

Transitive, because if $(x, y) \in R$ and $(y, z) \in R$, then $x - y = 3m$, for some integer m, and $y - z = 3n$, for some integer n. Hence, $(x - y) + (y - z) = 3m + 3n = 3(m + n)$. Thus, $x - z = 3(m + n)$. Notice that $m + n \in \mathbb{Z}$. This implies $x - z$ is divisible by 3. Therefore, $(x, z) \in R$.

An equivalence relation, because reflexive, symmetric, and transitive. Thus, we can talk about equivalence classes. We'll do that at the end.

Not antisymmetric, because, for example, $(1, 4)$ and $(4, 1)$ are in R, and $1 \neq 4$.

Not a partial order, because not antisymmetric.

Now let find the equivalence classes:

$$\begin{aligned} [0] &= \{x \in \mathbb{Z} \mid (x, 0) \in R\} \\ &= \{x \in \mathbb{Z} \mid x - 0 = 3m,\ m \in \mathbb{Z}\} \\ &= \{x \in \mathbb{Z} \mid x = 3m,\ m \in \mathbb{Z}\} \end{aligned}$$

Thus, [0] consists of all multiples of 3.

$$\begin{aligned} [1] &= \{x \in \mathbb{Z} \mid (x, 1) \in R\} \\ &= \{x \in \mathbb{Z} \mid x - 1 = 3m,\ m \in \mathbb{Z}\} \\ &= \{x \in \mathbb{Z} \mid x = 3m + 1,\ m \in \mathbb{Z}\} \end{aligned}$$

Thus, [1] consists of all integers whose remainder when divided by 3 is 1.

$$\begin{aligned} [2] &= \{x \in \mathbb{Z} \mid (x, 2) \in R\} \\ &= \{x \in \mathbb{Z} \mid x - 2 = 3m,\ m \in \mathbb{Z}\} \\ &= \{x \in \mathbb{Z} \mid x = 3m + 2,\ m \in \mathbb{Z}\} \end{aligned}$$

Thus, [2] consists of all integers whose remainder when divided by 3 is 2.

Since the union of the previous 3 classes is $\mathbb{Z}$ and since they are disjoint, then the equivalnce classes of R are [0], [1], [2], which are described above. Try to find $[-1]$, $[4]$, $[-2]$, $[5]$, $[-3]$, $[6]$, etc.

19. $A = \mathbb{Z}$, $\mathcal{R} = \{(x, y) \in \mathbb{Z} \times \mathbb{Z} \mid x - y < 3\}$.

20. $A = \mathbb{R}$. $(x, y) \in R$ iff $x - y \leq 3$.

21. A= set of all lines in the plane. $\mathcal{R} = \{(L_1, L_2) \mid L_1$ perpendicular to $L_2\}$.

22. $A = \mathbb{R}^2$, $\mathcal{R} = \{((a,b),(c,d)) \in \mathbb{R}^2 \times \mathbb{R}^2 \mid a - c = b - d\}$.

23. $A = \mathbb{R}^2$, $\mathcal{R} = \{((a,b),(c,d)) \in \mathbb{R}^2 \times \mathbb{R}^2 \mid ab = cd\}$.

24. $A = \mathbb{R}^2$, $\mathcal{R} = \{((a,b),(c,d)) \in \mathbb{R}^2 \times \mathbb{R}^2 \mid a = c, a \leq c\}$.

25. $A = \mathbb{R}$. $(x,y) \in R$ iff $x^2 + y^2 = 4$.

26. $A = \mathbb{R}$. $(x,y) \in R$ iff $x^2 + y^2 > 0$.

    Solution:

    Not reflexive, because $(0,0) \notin R$.

    Not an equivalence relation, because not reflexive.

    Not a partial order, because not reflexive.

    Symmetric, because if $(x,y) \in R$, then $x^2 + y^2 > 0$. Hence, $y^2 + x^2 > 0$. Therefore $(y,x) \in R$.

    Not transitive, because, for example, $(0,2) \in R$ and $(2,0) \in R$, and $(0,0) \notin R$.

    Not antisymmetric, because, for example, $(1,2) \in R$ and $(2,1) \in R$, and $1 \neq 2$.

27. $A = \mathbb{R} - \{0\}$. $(x,y) \in R$ iff $x^2 + y^2 > 0$.

28. $A = \mathbb{R}$. $(x,y) \in R$ iff $x^2 \geq y$.

    Solution: Done in class.

29. $A = \{1,2,3,4,5,6,7\}$. $(x,y) \in R$ iff xy is a perfect square.

30. $A = \mathbb{R}$.

$$R = \{(x,y) \in \mathbb{R} \times \mathbb{R} \mid x \leq y\}.$$

31. $A = \{1,2,3\}$. $R = \{(1,1),(2,2),(3,3),(1,2)\}$.

32. $A = \{1,2,3\}$. $R = \{(1,1),(2,2),(3,3),(1,2),(2,1)\}$.

33. $A = \{1,2,3\}$. $R = \{(1,1),(2,2),(3,3),(1,2),(2,1),(3,1)\}$.

34. $A = \{1,2,3\}$. $R = \{(1,1),(2,2)\}$.

(2) Let R be a binary relation on a nonempty set A. Prove or disprove:

1. R is reflexive and circular iff R is an equivalence relation.
2. $RoR^{-1} = R^{-1}oR$.

3. $|RoR^{-1}| = |R^{-1}oR|$.

4. RoR^{-1} and $R^{-1}oR$ are both symmetric.

5. If R is symmetric, then $R^{-1} = R$.

6. $RoR = R$.

7. RoR^{-1} is transitive.

8. RoR^{-1} is reflexive.

(3) Let R_1 and R_2 be equivalence relations on a nonempty set A. Prove or disprove:

1. $R_1 \cap R_2$ is an equivalence relation on A.

2. $R_1 \cup R_2$ is an equivalence relation on A.