

CSIT 241 Binary Relations

Definition: Let A and B be nonempty sets. A *binary relation* from A to B is a subset of $A \times B$.

Definition: Let A be a nonempty set. A *binary relation* on A is a subset of $A \times A$.

Remarks:

(1) If (x, y) belongs to a relation R , then we write xRy . Note that if xRy , then it is not necessarily that yRx .

(2) Every binary relation on a set A can be represented by a matrix.

(3) Every binary relation on a set A can be represented by a directed graph.

(4) Every function is a binary relation, but not every binary relation is a function.

Note that a binary relation on A simply defines a relation between the elements of A .

Example: Let A be the set of all residents of San Francisco. Define the following relation M on A :

$$M = \{(x, y) \mid x \text{ is the wife of } y\}.$$

Example: Let A be the set of all residents of San Francisco. Define the following relation *Couples* on A :

$$\text{Couples} = \{(x, y) \mid x \text{ is married to } y\}.$$

Example: Let A be the set of all humans. Define the following relation *EyeColor* on A :

$$\textit{EyeColor} = \{(x, y) \mid x \text{ has the same eye color as } y\}.$$

Example: Let A be the set of SUNY Fredonia's students and let B be the set of courses offered by SUNY Fredonia. Define the following relation *SameCourse* from A to B :

$$\textit{SameCourse} = \{(s, c) \mid s \text{ is taking } c\}.$$

Example: Let $A = \mathbb{R}$. Define the following relation S on A :

$$S = \{(x, y) \in \mathbb{R} \times \mathbb{R} \mid x \leq y\}.$$

Example: Let $A = \mathbb{Z}$. Define the following relation T on A :

$$T = \{(x, y) \in \mathbb{Z} \times \mathbb{Z} \mid x^2 = y^2\}.$$

Definitions: Let R be a relation on a set A . R is said to be

1. *Reflexive* if $(a, a) \in R$, for all $a \in A$.
2. *Symmetric* if $(a, b) \in R$ implies $(b, a) \in R$, for all a and b in A .
3. *Transitive* if $(a, b) \in R$ and $(b, c) \in R$, implies $(a, c) \in R$, for all a , b , and c in A .
4. *Antisymmetric* if $(a, b) \in R$ and $(b, a) \in R$ implies $a = b$, for all a and b in A . I.e. if $a \neq b$ and $(a, b) \in R$, then (b, a) cannot be in R . If $b = a$, then there is no problem.
5. *An equivalence relation* if R is reflexive, symmetric, and transitive.

6. A *partial order* if R is reflexive, antisymmetric, and transitive.
7. asymmetric if whenever $(a, b) \in R$, then $(b, a) \notin R$, for all a and b in A .
8. *circular* if $(a, b) \in R$ and $(b, c) \in R$, implies $(c, a) \in R$, for all a, b , and c in A .

Example: The relation S defined above is reflexive, because every real number is less than or equal to itself. Mathematically, $x \leq x$, for all $x \in \mathbb{R}$. Thus, $(x, x) \in S$, for all $x \in \mathbb{R}$.

S is not symmetric. For example, $(2, 3) \in S$, because $2 \leq 3$. But, $(3, 2)$ is not in S , because 3 is not less than or equal to 2.

S is transitive because if $(x, y) \in S$ and $(y, z) \in S$, then $(x, z) \in S$. Because $(x, y) \in S$ and $(y, z) \in S$ mean $x \leq y$ and $y \leq z$. Thus, $x \leq y \leq z$. Therefore, $x \leq z$, which implies $(x, z) \in S$.

S is antisymmetric because if $(x, y) \in S$ and $(y, x) \in S$, then $x \leq y$ and $y \leq x$. Thus, $x \leq y \leq x$. This implies $y = x$.

Thus, S is a partial order but not an equivalence relation.

Example: The relation T defined above is reflexive because every real number is equal to the square of itself. Mathematically, $x^2 = x^2$, for all $x \in \mathbb{Z}$. Thus, $(x, x) \in T$, for all $x \in \mathbb{Z}$.

T is symmetric, because if $(x, y) \in T$, then $x^2 = y^2$. This implies $y^2 = x^2$, which implies $(y, x) \in T$.

T is transitive because if $(x, y) \in T$ and $(y, z) \in T$, then $(x, z) \in T$. Because $(x, y) \in T$ and $(y, z) \in T$ mean $x^2 = y^2$ and $y^2 = z^2$. Thus, $x^2 = y^2 = z^2$. Therefore, $x^2 = z^2$, which implies $(x, z) \in T$.

T is not antisymmetric. For example, $(-2, 2) \in T$ and $(2, -2) \in T$, but $2 \neq -2$.

Thus, T is an equivalence relation but not a partial order.

Let R be the relation on \mathbb{Z} defined by:

$$aRb \text{ iff } 3a + b \text{ is a multiple of } 4.$$

Is R an equivalence relation on \mathbb{Z} ?

Solution: R is an equivalence relation, because

(1) R is reflexive, because $\forall a \in \mathbb{Z}$, $3a + a = 4a$ which is divisible by 4. Thus, $(a, a) \in R$.

(2) R is symmetric, because if $(a, b) \in R$, then $3a + b = 4m$, for some $m \in \mathbb{Z}$. Thus, $b = 4m - 3a$. But now $3b + a = 3(4m - 3a) + a = 4(3m - 2a)$. Notice that $3m - 2a$ is an integer. Thus, $3b + a$ is divisible by 4, which means that $(b, a) \in R$.

(3) R is transitive, because if (a, b) and (b, c) are in R , then $3a + b = 4m$ for some $m \in \mathbb{Z}$ and $3b + c = 4k$ for some $k \in \mathbb{Z}$. Therefore, $3a + c = 4m - b + 4k - 3b = 4(m + k - b)$. Notice that $m + k - b$ is an integer. Thus, $3a + c$ is divisible by 4, which means $(a, c) \in R$.

Definition: Let R be an equivalence relation on a set A and let $a \in A$. The *equivalence class* of a is defined to be all elements of A that are related to a . The equivalence class of a is denoted $[a]$ or \bar{a} . Mathematically,

$$[a] = \{b \in A \mid (a, b) \in R\}.$$

Note: To find $[a]$ in R , look for all ordered pairs in R that start with a .

Definition: Let A and B be sets. A and B are said to be disjoint if $A \cap B = \phi$.

Definition: Let A_1, A_2, \dots, A_n be nonempty sets. A_1, A_2, \dots, A_n are said to be **pairwise disjoint** if the intersection of every two distinct ones of them is empty.

Definition: Let A_1, A_2, \dots, A_n be nonempty subsets of A . A_1, A_2, \dots, A_n are said to **partition** A if they are pairwise disjoint and if $A_1 \cup A_2 \cup \dots \cup A_n = A$.

Facts:

(1) Let \mathcal{R} be an equivalence relation on a set A . Any two equivalence classes of \mathcal{R} are either equal or disjoint. The union of all equivalence classes is equal to A and the distinct equivalence classes partition A .

(2) If $(a, b) \in R$, then $[a] = [b]$.

Remarks:

(1) Equivalence classes are defined only for equivalence rela-

tions.

(2) Let \mathcal{R} be a partial order on A , then \mathcal{R} establishes an order among the elements of A . One way to represent partial orders is Hasse diagrams.

Remark: Symmetric and antisymmetric are not opposite of each other, which means not symmetric does not mean antisymmetric and not antisymmetric does not mean symmetric. This means a relation can be both symmetric and antisymmetric at the same time. For example, if $A = \{1, 2, 3\}$ and $R_1 = \{(2, 2)\}$. Then R_1 is both symmetric and antisymmetric. R_1 is also transitive, but it is not reflexive because (for example) $(3, 3)$ is not in R_1 . So, R_1 is neither an equivalence relation nor a partial order. Now let $R_2 = \{(1, 1), (2, 2), (3, 3)\}$ be a relation on the set A defined above. Then R_2 is an equivalence relation and a partial order. Actually, R_2 is the only relation on A which is both an equivalence relation and a partial order.

Example 0.1 Let $A = \{1, 2, 3\}$ and $\mathcal{R} = \{(1, 1), (2, 2), (3, 3), (1, 2), (2, 1)\}$. Then \mathcal{R} is an equivalence relation on A . The equivalence classes are:

$$[1] = \{ \text{all elements related to } 1 \} = \{1, 2\}.$$

$$[2] = \{ \text{all elements related to } 2 \} = \{2, 1\}.$$

So, $[2] = [1]$. Remember the order within the set is not important.

$$[3] = \{ \text{all elements related to } 3 \} = \{3\}.$$

Notice that the equivalence class of 3 is disjoint from the equivalence class of 1 and from the equivalence class of 2. Notice also that the union of the equivalence classes is equal to A .

Question: Is it possible to have a binary relation which is both equivalence and a partial order? If yes, then give an example and explain the nature of such a binary relation.

Answer: Let A be any nonempty set and let R be a binary relation on A . In order for R to be both equivalence and a partial order, we need R to be both reflexive, transitive, symmetric, and antisymmetric. But, in order for R to be antisymmetric, if it has an element of the form (a, b) , and if $a \neq b$, then it cannot have (b, a) . But, if (a, b) is in R and (b, a) is not, then R becomes not symmetric. Thus, R cannot have elements of the form (a, b) if $a \neq b$. Thus, the only elements which R can have are those of the form (a, a) . But, in order for R to be reflexive, we need (a, a) to be in R for every element $a \in A$. (Notice that we have not used transitivity.) For example, if $A = \{1, 2, 3\}$, then the only equivalence relation and partial order on A is $\{(1, 1), (2, 2), (3, 3)\}$.

Composition of Binary Relations

Definition 0.2 Let R_1 be a binary relation from X to Y and R_2 a binary relation from Y to Z ; then

$$R_2 \circ R_1 = \{(x, z) \mid (x, y) \in R_1 \text{ and } (y, z) \in R_2 \text{ for some } y \in Y \}.$$

Definition 0.3 Let R be a relation on X ; then

$$R^{-1} = \{(x, y) \mid (y, x) \in R\}.$$

Question: Let R_1 be a binary relation from X to Y and R_2 a binary relation from Y to Z . Prove or disprove: $R_1 \circ R_2 = R_2 \circ R_1$.

Solution: The statement is false. You can come up with a counterexample easily. Do not forget that

$R_1 \circ R_2$ may be even undefined.

Remark 0.1 Notice that every function is a relation. In other words, functions are special cases of relations. For example, if f is the function from $X = \{-1, 0, 2\}$ onto $Y = \{0, 1, 4\}$ given by $f(x) = x^2$. Then f can be represented as $f = \{(-1, 1), (0, 0), (2, 4)\}$.

Notice also that if f is a function from X into X and if f^{-1} exists, then $f \circ f^{-1}(x) = f^{-1} \circ f(x) = x, \forall x \in X$. As a relation, we have $f \circ f^{-1} = f^{-1} \circ f = \{(x, x) \mid x \in X\}$. The question now is: does the same thing hold for any relation R on a set X ? If no, then what are the properties of $R \circ R^{-1}$? We'll address some of those properties shortly.

Question: Let R be a relation on X . Prove or disprove:

- (1) $R \circ R^{-1} = R^{-1} \circ R$.
- (2) $|R \circ R^{-1}| = |R^{-1} \circ R|$.
- (3) $R \circ R^{-1}$ and $R^{-1} \circ R$ are both symmetric.
- (4) If R is symmetric, then $R^{-1} = R$.
- (5) $R \circ R = R$.

Solution:

- (1) False. Counterexample: take $X = \{1, 2, 3\}$ and $R = \{(1, 3)\}$. Then $R^{-1} = \{(3, 1)\}$. Hence, $R \circ R^{-1} = \{(3, 3)\}$ and $R^{-1} \circ R = \{(1, 1)\}$.
- (2) False. Counterexample: take $X = \{1, 2, 3\}$ and $R = \{(1, 2), (3, 2)\}$. Then $R^{-1} = \{(2, 1), (2, 3)\}$. Hence, $R \circ R^{-1} = \{(2, 2)\}$ and $R^{-1} \circ R = \{(1, 1), (1, 3), (3, 1), (3, 3)\}$.
- (3) True. The proof was done in class.
- (4) Straightforward.
- (5) Easy.

Question: Let R be a relation on X . Prove or disprove:

- (1) $R \circ R^{-1}$ is transitive.
- (2) $R \circ R^{-1}$ is reflexive.

Question: Let R be the relation on \mathbb{Z} defined by:

$$aRb \text{ iff } 3a + b \text{ is a multiple of 4.}$$

Is R an equivalence relation on \mathbb{Z} ? If yes, then what is [1]. Also, find R^{-1} .

Solution: R is an equivalence relation, because

(1) R is reflexive, because $\forall a \in \mathbb{Z}, 3a+a = 4a$ which is divisible by 4. Thus, $(a, a) \in R$.

(2) R is symmetric, because if $(a, b) \in R$, then $3a + b = 4m$, for some $m \in \mathbb{Z}$. Thus, $b = 4m - 3a$. But now $3b + a = 3(4m - 3a) + a = 4(3m - 2a)$. Notice that $3m - 2a$ is an integer. Thus, $3b + a$ is divisible by 4, which means that $(b, a) \in R$.

(3) R is transitive, because if (a, b) and (b, c) are in R , then $3a + b = 4m$ for some $m \in \mathbb{Z}$ and $3b + c = 4k$ for some $k \in \mathbb{Z}$. Therefore, $3a + c = 4m - b + 4k - 3b = 4(m + k - b)$. Notice that $m + k - b$ is an integer. Thus, $3a + c$ is divisible by 4, which means $(a, c) \in R$.

$$\begin{aligned}
 [1] &= \{b \in \mathbb{Z} \mid aR1\} \\
 &= \{a \in \mathbb{Z} \mid 3a + 1 = 4k, k \in \mathbb{Z}\} \\
 &= \{a \in \mathbb{Z} \mid a = \frac{4k - 1}{3}, k \in \mathbb{Z}\} \\
 &= \{a \in \mathbb{Z} \mid a = 4k - 3, k \in \mathbb{Z}\}
 \end{aligned}$$

$$R^{-1} = \{(a, b) \mid a + 3b \text{ is a multiple of } 4\}.$$

Question: Define the following relation on R^2 :

$$(a, b)R(c, d) \text{ iff } a^2 + b^2 = c^2 + d^2.$$

Is R an equivalence relation? If yes, then what is the equivalence class of (α, β) and what does it represent in the Cartesian plane? Also, find R^{-1} .

Solution: R is an equivalence relation. This can be easily verified. Also, it is easy to see that $R^{-1} = R$. Why?

Now, $[(\alpha, \beta)] = \{(x, y) \mid x^2 + y^2 = \alpha^2 + \beta^2\}$. It is easy to see that $[(\alpha, \beta)]$ is a circle centered at the origin and with radius $\sqrt{\alpha^2 + \beta^2}$. Actually, the disjoint equivalence classes partition the Cartesian plane (i.e. \mathbb{R}^2) into a set of circles centered at the origin.