Question 1:

(a) $D - C = \{(1,1),(1,-1),(-1,\frac{1}{2}),(1,\frac{1}{2}),(-1,1),(-1,-1),(\frac{1}{2},1),(\frac{1}{2},-1)\}$.

(b) $F \cap D = \{(\frac{1}{2},\frac{1}{2})$.

Question 2:

(a) The function $f(x) = 2x$ if $x \in \mathbb{Z}^+$ and $f(x) = 1 - 2x$ if $x \in Z^- \cup \{0\}$ is a bijective function from $\mathbb{Z}$ onto $\mathbb{N}$. You have to prove that it is bijective or you can depend on the following

- The function $g : \mathbb{N} \longrightarrow \mathbb{Z}$, defined by $g(x) = x/2$ if x is even, and $g(x) = (1-x)/2$ if x is odd, is bijective.
- If $f : A \longrightarrow B$ is bijective, then so is $f^{-1} : B \longrightarrow A$.

(b) We want to prove that the function $f : \mathbb{N} \longrightarrow [e^{\frac{1}{4}}, e)$ defined by $f(x) = e^{\frac{x}{x+3}}$ is one-to-one.

Let x_1 and x_2 be in $\mathbb{N}$ and assume that $f(x_1) = f(x_2)$. Then $e^{\frac{x_1}{x_1+3}} = e^{\frac{x_2}{x_2+3}}$. Now take the natural logarithm of both sides to get that $\frac{x_1}{x_1+3} = \frac{x_2}{x_2+3}$. By cross multiplying, we get $x_1(x_2+3) = x_2(x_1+3)$. Thus, $x_1x_2 + 3x_1 = x_2x_1 + 3x_2$. Hence, $x_1 = x_2$.

Question 3:

(a) Let R be the relation on $\mathbb{Z}$ defined by

$$a \; R \; b \text{ if and only if } a = |b|^3.$$

Is R antisymmetric? Is is transitive? If the answer to any of them is yes, prove that. If the answer is not, then write down a counter example. Also, find R^{-1}.

R is antisymmetric. To prove that use the following definition which I gave in class:

$$R \text{ is antisymmetric iff whenever } (a,b) \text{ and } (b,a) \text{ are in } R, \text{ then } a = b.$$

So, assume that (a, b) and (b, a) are in R for some a, b in $\mathbb{Z}$. Now $(a, b) \in R$ implies that $a = |b|^3$ and $(b, a) \in R$ implies that $b = |a|^3$. Thus, $a = |a|^9$. But, $|a|^9$ is nonegative. Hence, a is nonnegative. Thus, we can get rid of the absolute value, to get $a = a^9$. Therefore, either $a = 0$ or $a = 1$. If $a = 0$, then $b = 0$ and if $a = 1$, then $b = 1$.

R is not transitive, because, for example, $(512, 8)$ and $(8, 2)$ are both in R, but $(512, 2)$ is not. Notice that $512 \neq 2^3$.

Notice that $R^{-1} = \{(a, b) \mid b = |a|^3\}$.

(b) The relation R on $\mathbb{Z}$ defined by

$$a \ R \ b \text{ if and only if } 3 - b \leq a - 2b < 1 + b.$$

is not transitive, because, for example, $(10, 5)$ and $(5, 2)$ are both in R, but $(10, 2)$ is not.

R is not symmetric, because, for example, $(10, 5) \in R$, but $(5, 10)$ is not.

(c) For the relation R on $\mathbb{Z}$ defined by

$$a \ R \ b \text{ if and only if } a - b \text{ is divisible by } 5.$$

$$\begin{aligned} [17] &= \{a \in \mathbb{Z} \mid aR17\} \\ &= \{a \in \mathbb{Z} \mid a - 17 = 5k, \ k \in \mathbb{Z}\} \\ &= \{a \in \mathbb{Z} \mid a = 17 + 5k, \ k \in \mathbb{Z}\} \\ &= \{a \in \mathbb{Z} \mid a = 2 + 5m, \ m \in \mathbb{Z}\}. \end{aligned}$$

Question 5: Let $\{a_n\}_{n=1}^{\infty}$ be defined as follows:

$$a_n = 1 + (-1)^n \text{ if } n \text{ is odd and } a_n = (n - 1) + n \text{ if } n \text{ is even.}$$

Let $S_n = \sum_{k=1}^{n} a_k$.

(a) The general formula for S_n, $n \in \mathbb{N}$, is:

$S_n = \frac{n(n+1)}{2}$ if n is even and $S_n = \frac{n(n-1)}{2}$ if n is odd.