Graphs Practise Questions II

In the following, assume all graphs are simple and undirected.

Question 1: Prove that the only cyclic graph isomorphic to its complement is C_5.

Solution: **First, show that C_5 is isomorphic to its complement by finding an isomorphism between them. That can be easily done and I'll leave it for you.**

Now let prove that if $n \neq 5$, then C_n is not isomorphic to its complement. The proof is quite easy. It depends on the fact that if C_n is isomorphic to its complement, then C_n and its complement must have the same number of edges. But the number of edges of C_n is n and the number of edges of $\overline{C_n}$ is $\frac{n(n-1)}{2} - n$. Thus, in order for C_n to be isomorphic to $\overline{C_n}$, we need $n = \frac{n(n-1)}{2} - n$. Solve this equation to get $n = 0$ or $n = 5$. Of course, $n = 0$ is rejected. So, it must be that $n = 5$.

Question 2: Let $G = (V, E)$ be undirected graph, prove that if G is isomorphic to its complement, then either $|V|$ or $|V| - 1$ is a multiple of 4.

Solution: **The proof depends on the fact that if G is isomorphic to its complement, then G and its complement must have the same number of edges. But the number of edges of G is $|E|$ and the number of edges of $\overline{G}$ is $\frac{n(n-1)}{2} - |E|$, where $n = |V|$. Thus, in order for G to be isomorphic to $\overline{G}$, we need $|E| = \frac{n(n-1)}{2} - |E|$. Thus, $2|E| = \frac{n(n-1)}{2}$, which implies $n(n-1) = 4|E|$. Now notices that $|E|$ is a nonnegative integer. Also, notice that both n and $n - 1$ are nonnegative integers and that if n is even, then $n - 1$ is odd and vice versa. Thus, in order for $n(n-1)$ to be a multiple of 4, it must be that one of them (I mean n and $n - 1$) is a multiple of 4.**

Question 3: Prove that if the bipartite $G_1 = (V_1, E_1)$ is isomorphic to $G_2 = (V_2, E_2)$, then G_2 is also bipartite.

Question 4: Let S be a clique in $G_1 = (V_1, E_1)$, and assume that G_1 is isomorphic to $G_2 = (V_2, E_2)$ with an isomorphism f. Is $f(S)$ a clique in G_2?

Question 5: Prove that if $G = (V, E)$ is a complete bipartite graph that is isomorphic to $H = (V', E')$, then H is also a complete bipartite graph.

Solution: **Since $G = (V, E)$ is a complete bipartite graph, then V can be partitioned into two disjoint sets V_1 and V_2 whose union is V and such that every vertex of V_1 is adjacent to every vertex of V_2 and no two vertices of V_1 are adjacent and no two vertices of V_2 are adjacent. Now let f be an isomorphism from V onto V'. Then, I claim that $f(V_1)$ and $f(V_2)$ form a partition of $V' = f(V)$ and**

1. **Every vertex of $f(V_1)$ is adjacent to every vertex of $f(V_2)$.**
2. **No two vertices of $f(V_1)$ are adjacent.**
3. **No two vertices of $f(V_2)$ are adjacent.**

Notice that $f(V_1) \cap f(V_2) = \phi$, because f is one-to-one and V_1 and V_2 are disjoint. Notice also that $f(V_1) \cup f(V_2) = f(V) = V'$, because f is onto and $V_1 \cup V_2 = V$. Now let me prove the above three claims:

1. **Let $c \in f(V_1)$ and $d \in f(V_2)$. Then, there exist $a \in V_1$ and $b \in V_2$ such that $f(a) = c$ and $f(b) = d$. Now since (a, b) is an edge in G (because every vertex in V_1 is adjacent to every vertex in V_2) and since f is an isomorphism, then it follows that $(f(a), f(b))$ is an edge in H. Thus, (c, d) is an edge in H. Therefore, c and d are adjacent.**
2. **The proof is by contradiction. Assume that there are two vertices in $f(V_1)$ that are adjacent in H. Let these two vertices be α and β. But, now f is onto implies that there exist a and b in V_1 such that $f(a) = \alpha$ and $f(b) = \beta$. Now since f is an isomorphism and since α and β are adjacent in H, then it follows that a and b are adjacent in G. This contradicts the fact that no two vertices of V_1 are adjacent.**
3. **Similar to the previous one.**

Question 6: Give four different proofs for the identity:

$$1 + 2 + 3 + ... + n = \frac{n(n+1)}{2}, \ \forall n \in \mathbb{N}.$$

One of your proofs should be based on the number of edges of K_m (the complete graph on m vertices).

Solution:

First Proof: **By induction.**

Second Proof: **Use the identity**

$$C(n+1, k+1) = \sum_{i=k}^{n} C(i, k), \ \forall n, k \in \mathbb{N}.$$

Now let $k = 1$ to get:

$$C(n+1, 2) = \sum_{i=1}^{n} C(i, 1), \ \forall n \in \mathbb{N}.$$

Thus,

$$\frac{(n+1)!}{2! \cdot (n-1)!} = C(1, 1) + C(2, 1) + C(3, 1) + ... + C(n-1, 1) + C(n, 1).$$

Hence,

$$\frac{n(n+1)}{2} = 1 + 2 + 3 + ... + (n-1) + n.$$

Third Proof: **Notice that $1, 2, 3, ..., n$ is an arithmetic sequence. Notice also that the sum of an arithmetic sequence is $\frac{n}{2} \cdot [2a + (n-1)d]$, where a is the first term of the sequence and d is the difference between an two succesive terms. Thusm in our case, $a = 1$ and $d = 1$. Therefore, the sum of our sequence is $\frac{n}{2} \cdot [2 + (n-1)] = \frac{n(n+1)}{2}$.**

Fourth Proof: **Here we depend on the number of edges of the complete graph on m vertices K_m, which is $\frac{m(m-1)}{2}$. Also, we depend on the fact**

The number of edges of $K_{n+1} = n +$ **the number of edges of** K_n**.**

Thus,

$$\frac{(n+1)n}{2} = n + (n-1) + \text{the number of edges of } K_{n-1}.$$

Now keep going up to K_2**, to get**

$$\frac{(n+1)n}{2} = n + (n-1) + (n-2) + ... + 2 + 1.$$

which is the desired result.

Question 7: All examples we did in class.